Title:
Curve Modeling with Jacobi Elliptic Functions

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Introduction:
In this study, we propose to use Jacobi elliptic functions as blending functions for free-form curve formulations. For example, it is known that the shape of a rotating rope can be expressed by elliptic functions, and it is meaningful to use elliptic functions as blending functions. We propose to use elliptic functions as blending functions, following the construction method of basis functions for Multiquadratic Curves: MQ-Curves, which is an Extended Complete Tchebycheff System.

Multiquadratic Curves: MQ-Curves:
Multiquadratic(MQ-)curve [2] uses the space spanned by

$$U = \{1, t, \sqrt{c^2 + t^2}, \sqrt{c^2 + (1-t)^2}\}, \quad c \neq 0, t \in I = [0, 1]. \tag{2.1}$$

The normal curve $u \in A^3$ is an arc of an algebraic curve of order 4, since u lies in the intersection of the two hyperbolic cylinders $x_2^2 - c_1^2 = c^2$ and $x_3^2 - (1 - x_1)^2 = c^2$. Eck [2] presented a Bézier-like representation of MQ-curves and derived some interesting properties. We have now an easy approach to MQ-curves. The curves possess the convex hull property and the variation diminishing property which respect to their control polygon $b_0b_1b_2b_3$.

Local Basis Function:
Based on Eck [2] in this subsection, we will think about MQ-curve segment. It is defined as

$$\phi_0(c, t) = a_0(1 - t) + a_1\sqrt{c^2 + (1 - t)^2} + a_2\sqrt{c^2 + t^2} + a_3 t \tag{2.2}$$

They introduced a complicated local blending functions for a MQ-curve segment with properties similar to the Bernstein basis in the case of polynomials. Eq.(2.2) is written as follows

$$\phi(c, t) = \sum_{i=0}^{3} b_i \Omega_i(c, t) \quad t \in [0, 1] \tag{2.3}$$

$\Omega_i(c,t)$ are defined

$$\begin{aligned}
&\Omega_1(c,t) = (s(c)+c)^2\frac{\beta(c)}{\alpha(c)}(t - s(c)\sqrt{c^2+(1-t)^2} + c\sqrt{c^2+(t^2)})\\
&\Omega_2(c,t) = \omega_1(c,1-t)\\
&\Omega_3(c,t) = t - \alpha(c)\Omega_1(c,t) + (\alpha(c)-1)\Omega_2(c,t)\\
&\Omega_0(c,t) = \Omega_(c,1-t)
\end{aligned} \tag{2.4}$$

where the auxiliary functions used in the above equations are defined as

$$s(c) = \sqrt{c^2+1}, \quad \beta(c) = \frac{s(c)}{s(c)+c}, \quad \alpha(c) = \frac{cs(c)}{2c^2+1+cs(c)} \tag{2.5}$$

In Eq.(2.3) the control points b_i is determined by

$$\begin{pmatrix} b_0\\ b_1\\ b_2\\ b_3 \end{pmatrix} = \begin{pmatrix} 1 & s(c) & c & 0\\ 1-\alpha(c) & s(c)-\frac{\alpha(c)}{s(c)} & c & \alpha(c)\\ \alpha(c) & c & s(c)-\frac{\alpha(c)}{s(c)} & 1-\alpha(c)\\ 0 & c & s(c) & 1 \end{pmatrix} \begin{pmatrix} a_0\\ a_1\\ a_2\\ a_3 \end{pmatrix} \tag{2.6}$$

The relations among the control points and Eq.(2.3) are

$$\begin{aligned}
&b_0 = \phi_0(c,0)\\
&b_1 = \phi_0(c,0) + \alpha(c)\phi_0'(c,0) = \phi_0(c,1) + (\alpha(c)-1)\phi_0'(c,1) + \alpha(c)\beta(c)\phi_0''(c,1)\\
&b_2 = \phi_0(c,1) - \alpha(c)\phi'(c,1) = \phi_0(c,0) + (1-\alpha(c))\phi_0'(c,0) + \alpha(c)\beta(c)\phi_0''(c,0)\\
&b_3 = \phi_0(c,1)
\end{aligned} \tag{2.7}$$

where ' denotes differentiation of $\phi(c,t)$ with respect to t. Note that $\sum_{i=0}^{3}\Omega_i(c,t) = 1$ for $t \in [0,1]$.

No explanation is done on how to derive $\Omega_i(c,t)$ in Eck [2]. Therefore we will derive these blending functions and extend them. From Eq.(2.3),

$$\begin{aligned}
&b_1 - b_0 = \alpha(c)\phi_0'(c,0)\\
&b_3 - b2 = \alpha(c)\phi_0'(c,0)
\end{aligned} \tag{2.8}$$

The above equations means that the tangent vectors of a MQ-curve segment are parallel with $b_1 - b_0$ and $b_3 - b_2$ at the start point and end points just like the cubic Bézier curve. Eck [2] determine the coefficient of $\alpha(c)$ such that the curve in at Least three dimensions, b_1 is in fact the intersection points of the tangent in $\phi(c,0)$ and the osculating plane in $\phi(c,1)$, and vice versa for b_2.

From Eq(2.4),

$$\frac{\partial\Omega_0}{\partial t}(c,1) = 0, \quad \frac{\partial\Omega_1}{\partial t}(c,1) = 0 \tag{2.9}$$

Because of symmetry between $\Omega_0(c,t)$ and $\Omega_3(c,t)$ along $t = 1/2$ and $\Omega_1(c,t)$ and $\Omega_2(c,t)$

$$\frac{\partial\Omega_2}{\partial t}(c,0) = 0, \quad \frac{\partial\Omega_3}{\partial t}(c,0) = 0 \tag{2.10}$$

Furthermore the following equations are satisfied:

$$
\begin{aligned}
\frac{\partial \phi_0(c,t)}{\partial t}|_{t=0} &= b_1 - b_0 \\
\frac{\partial \phi_0(c,t)}{\partial t}|_{t=1} &= b_3 - b_2
\end{aligned}
\tag{2.11}
$$

The above conditions are rewritten by using a_i

$$
\begin{aligned}
b_1 - b_0 &= \frac{c\left(\sqrt{c^2+1}(a_3-a_0)+a_1\right)}{c\left(\sqrt{c^2+1}+2c\right)+1} \\
&= \frac{c\sqrt{c^2+1}}{c\left(\sqrt{c^2+1}+2c\right)+1}\frac{\partial \phi(c,t)}{\partial t}|_{t=0} \\
b_3 - b_2 &= \frac{c\left(\sqrt{c^2+1}(a_3-a_0)+a_2\right)}{c\left(\sqrt{c^2+1}+2c\right)+1} \\
&= \frac{c\sqrt{c^2+1}}{c\left(\sqrt{c^2+1}+2c\right)+1}\frac{\partial \phi(c,t)}{\partial t}|_{t=1}
\end{aligned}
\tag{2.12}
$$

Derivation of b functions:
We assume that the blending functions $h_i(,ct)$ of the MS-segment are linear combinations of $f_0(c,t) = 1-t$, $f_1(c,t) = \sqrt{c^2+(1-t)^2}$, $f_2(c,t) = \sqrt{c^2+t^2}$ and $f(_3(c,t) = t$. Furthermore we assume $h_2(c,t) = h_1(c,1-t)$, $h_3(c,t) = h_1(c,1-t)$.

Then $h_0(c,t)$ and $h_1(c,t)$ are defined by

$$
\begin{aligned}
h_0(c,t) &= \sum_{i=0}^{3} a_i f_i(c,t) \\
h_1(c,t) &= \sum_{i=0}^{3} b_i f_i(c,t)
\end{aligned}
\tag{2.13}
$$

where a_i and b_i do not depend on either c or t and they are constants. Their number is equal to 8. The conditions on these functions are

$$
\begin{aligned}
&h_0(c,0) = 1, \quad h_0(c,1) = 0, \quad \frac{\partial h_0(c,t)}{\partial t}|_{t=1} = 0 \\
&h_1(c,0) = 0, \quad h_1(c,1) = 0, \quad \frac{\partial h_0(c,t)}{\partial t}|_{t=0} = -\frac{c\left(\sqrt{c^2+1}+2c\right)+1}{c\sqrt{c^2+1}}
\end{aligned}
\tag{2.14}
$$

The last equation should be satisfied from Eq.(eq:tan0). We need the following conditions for the partition of unity, which should not depend on either c or t and

$$
\begin{aligned}
a_0 + a_3 + b_0 + b_1 &= 1 \\
a_1 + a_2 + b_1 + b_2 &= 0
\end{aligned}
\tag{2.15}
$$

We have 9 constraints and the number of variables is 8. Fortunately, these conditions are linearly dependent, and we can omit one of them. By solving a system of linear equations for a_i and b_i, $i = 0, 3$, we obtain $\Omega_i(c,t)$ in Eck [2].

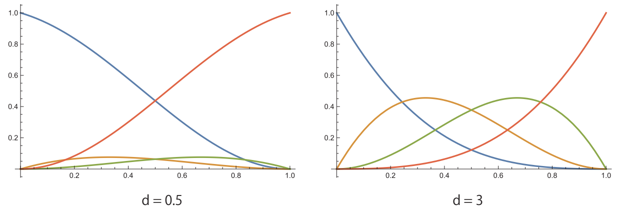

Fig. 1: New blending functions of the MQ-curve segment, $(c, d) = (1, 1/2)$ and $(c, d) = (1, 3)$.

New Blending Functions:

We can control the blending functions by changing the value $d = \frac{c(\sqrt{c^2+1}+2c)+1}{c\sqrt{c^2+1}}$ in Eq.(2.14). We will show blending functions with $d = 1/2$ and $d = 3$ in FIg.1.

Space spanned by $(t, \mathrm{sn}((1-t)K, k), \mathrm{sn}(Kt, k), t)$:

We would like to represent the shape of the rope by blending functions. Using $(t, \mathrm{sn}((1-t)K, k), \mathrm{sn}(Kt, k), t$ and adjusting parameters, the half shape of the rope is given by

$$\begin{aligned} C(t) &= (t, \mathrm{sn}(Kt, k)) \\ &= p_0(1-t) + p_1\mathrm{sn}((1-t)K, k) + p_2\mathrm{sn}(Kt, k), + p_3 t \end{aligned} \tag{2.16}$$

where $p_0(0, 0)$, $p_1 = (0, 0)$, $p_2 = (0, 1)$ and $p_3 = (1, 0)$.

Then the matric M_3 similar to M_0, $i = 0, 1, 2$ is given by

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -K(k) & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -K(k) & 0 & 1 \\ -1 & 0 & K(k) & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -d \\ 0 \end{pmatrix} \tag{2.17}$$

Therefore

$$\begin{aligned} (q_0, q_1, q_2, q_3) \begin{pmatrix} b_0(t) \\ b_1(t) \\ b_2(t) \\ b_3(t) \end{pmatrix} &= (q_0, q_1, q_2, q_3) M\, F \\ &= (p_0, p_1, p_2, p_3) F \end{aligned} \tag{2.18}$$

Then

$$(q_0, q_1, q_2, q_3) = (p_0, p_1, p_2, p_3) M^{-1} \tag{2.19}$$

For $(k, d) = (0.3, 2)$, we obtain $q_0 = (0, 0)$, $q_1 = (0.5, 0.856945)$, $q_2 = (0.5, 1)$ and $q_3 = (1, 1)$. Figure 2 show the locations of the control points and the curve generated with these control points to generate the half shape of the rope.

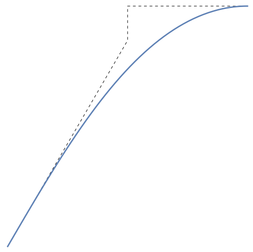

Fig. 2: The half shape of the rope

Conclusions:

In the full paper, we will propose to use Jacobi elliptic functions as blending functions for free-form curve formulations based on the method described in this extended abstract. For example, it is known that the shape of a rotating rope can be expressed by elliptic functions, and it is meaningful to use elliptic functions as blending functions. We has proposed to use elliptic functions as blending functions, following the construction method of basis functions for Multiquadratic Curves: MQ-Curves, which is an Extended Complete Tchebycheff System.

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