Title:
Least Squares Fitting on a Segment of Ellipse and Its Application on Road Curvature Estimation

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Introduction:

The fitting of geometric features such as circles and ellipses to given points is applicable in various fields [16]. Ellipse, for example, may represent many real-world situations, such as the orbits of the planets, satellites, and comets. These conic sections are also relevant in science, astronomy, and variety of engineering applications [6]. In real-world practice, least squares fitting is used in various manufacturing industries, software applications [18], and also significant in pattern recognition [3]. A circle, an ellipse, a parabola or a hyperbola are curves that are referred to as conic sections. In this paper, elliptical arcs are preferred over general conic arcs as ellipses are frequently encountered shapes found as a component in various objects and have been proven to be beneficial in fields such as Computer-Aided Design (CAD), computer graphics and computer vision [13]. For instance, we are interested in fitting data that form an arc or a segment of an ellipse. Hence, we apply an ellipse fitting rather than other curve fittings. In addition, ellipse ranks among the most prevalent geometric shapes found in the real world [17].

In this paper, we are interested to fit parametric curves in the least squares sense. The best fitting curve to a given set of points is obtained by minimizing the sum of the square errors of the points from the curve [1]. This mathematical procedure is called the least squares fitting. Gander et al. [8], Watson [18], and Pilu et al. [12] apply least squares fitting based on data that forms entire circles and ellipses but here we only discuss a part of an ellipse or a circle that represents an arc [4] or a curve. While the parameter z for a full ellipse ranges from 0 to 2π radians, the challenge with data that form a partial ellipse is to determine where it lies in the parameter z. We will determine the parameter z that best fits the data points presumed to be part of an ellipse. Then, we demonstrate how this parameter can be utilized to estimate road curvature and compare it with the experimental approach employed by engineers.

A simple algorithm that uses a numerical optimization technique is introduced to obtain the best curve fitting for any given set of data, specifically data that forms a segment of an ellipse. This technique is known as simulated annealing. There are extensive usage of simulated annealing in real-life applications in which the primary benefit of it lies in its simplicity [5]. In road curvature estimation, people may relate road curvature with other benefits. For example, Persyn et al. [11] consider various factors related to road characteristics referring to OpenStreetMap (OSM) to estimate the expenses associated with road transportation. The factors include the existence of roundabouts and traffic lights, the surface material and the curvature of the road. Good estimation of transportation costs is very helpful for trip planning, consequently may result in better fuel consumption and less pollution to the environment. The general parametric conic arcs introduced by [7] outline the process of creating conic blending arcs by utilizing a unified rational parametric representation that merges the distinct cases of blending parallel and non-parallel edges based on given constraints such that it requires the arc to maintain a specified distance from a line, point, or a circle. Otherwise, the arc intersects a circle or line at a specified angle. In this paper, instead of interpolating points, we used least squares fitting to approximate points for any given data.

Fitting an Ellipse in Parametric Form:

Generally, in order to fit an ellipse in parametric form, we follow Späth [14] and consider the equations:

$$\begin{aligned} x(z) &= a + p\ cos(z) \\ y(z) &= b + q\ sin(z) \end{aligned} \tag{2.1}$$

where (a, b) is the center of the ellipse, p is the radius along the x-axis, q is the radius along the y-axis and parameter z lies between 0 to 2π radians. The function to be minimized is:

$$S(a,b) = \sum_{i=0}^{n} (x_i - a - p\ \ cos(z_i))^2 + (y_i - b - q\ \ sin(z_i))^2, \tag{2.2}$$

where z_i is a parametrized value that lies between 0 to 2π radians and (x_i, y_i) are data points.

Späth [14] utilizes (2.1) to fit an ellipse based on data forming a complete elliptical shape. However, we consider the case where collected data are from a segment and not the entire ellipse. The issue is dealt with in [2]; however, we use a parametrization approach in minimization as described in the next section.

In this case, we aim to minimize (2.2) where z_i is a parametrized value that lies between θ_1 and θ_2. Parameter z_i should cover a certain part of an ellipse. For instance, if the data is half of an ellipse that forms the upper half of ellipse, z_i should cover from 0 to π radians. If let's say the data forms the bottom half of the ellipse, then z_i can be from π to 2π radians. The range can be determined through observation; nonetheless, we will select the optimal values for θ_1 and θ_2 by optimizing the equation (2.2).

The values of a, b, p and q can be solved by differentiating (2.2) with respect to each parameter and equate it to 0:

$$\frac{\delta S}{\delta a} = 0\ ,\ \frac{\delta S}{\delta b} = 0\ ,\ \frac{\delta S}{\delta p} = 0\ ,\ \frac{\delta S}{\delta q} = 0. \tag{2.3}$$

The presence of parameter z_i and the uncertainty regarding its interval render the problem difficult to solve. Therefore, we will determine the values of a, b, p, and q using simulated annealing, a method that will be further elaborated in the next section.

Minimizing the Error Distances:

To establish the minimization process using simulated annealing, we start by discussing about parameter z in the ellipse function. The parameter z_i can be computed by:

$$z_i = \theta_1 + (i-1)h, \tag{2.4}$$

where $i = 1, 2, ..., n$ and θ_1 denotes the start of the interval while h is defined as the step size for parameter z_i and it is assigned arbitrarily.

For instance, if $h = 0.1$, then $z = \{\theta_1, \theta_1 + 0.1, \theta_1 + 0.2, \theta_1 + 0.3, ...\}$. By choosing any value from θ_1 to θ_2 from this range, where $\theta_2 = \theta_1 + (n-1)h$, we can see the pattern of the error distance to be either increasing or decreasing. To minimize (2.2), we employ a numerical approach to evaluate the value of $S(a, b)$. We aim to find the values of θ_1 and h that will minimize the function.

The error distance can be obtained by using:

$$d = \sum_{i=0}^{n} |(X_i, Y_i) - (x_i, y_i)|, \tag{2.5}$$

where (X_i, Y_i) are the points on the estimated curve and (x_i, y_i) are the original data points. Hence the minimum value of (2.5) is the solution to the minimization problem.

We perform the minimization by using a numerical optimization technique called simulated annealing which is available as a built-in function on Mathematica. The purpose is to find the optimum values of z, and parameters a, b, p, q, θ_1 and h in order to obtain the best curve fitting of an ellipse. The next part of the algorithm is to input data i.e. number of observations, n, coordinates (x_1, y_1), (x_2, y_2),..., (x_n, y_n), v_1, and v_2. Then, minimizing (2.2) using simulated annealing subject to constraint $0 \leq \theta_1 \leq 2\pi$ and $v_1 \leq h \leq v_2$, whereas v_1 and v_2 are the minimum and maximum step sizes respectively. For our experiment, we let v_1 and v_2 ranges from 0.1 to 0.5 where $v_1 < v_2$. Meanwhile, the output obtained are the parameters a, b, p, and q for the equation of ellipse, the minimum error distance, and the values of θ_1 and h.

The Application of Ellipse Fitting on Road Curvature Estimation:

After the minimization procedure, during which we obtained the best-fitting ellipse for data forming a segment of an ellipse, we now aim to apply the proposed algorithm to fit data from small segments of roads, particularly those with curvy shapes. Few points will be taken along the desired segment of a road. Hence, fitting an ellipse on the segment of a road will allow us to calculate the radius of curvature for each point precisely based on its coordinate on the road.

For the radius of curvature, we compare our approach to Luo et al. [9]. In their paper, radius of curvature was calculated at 9 different test sites, chosen from highway ramps and field measurement were used to conduct the validation tests. Besides, this paper uses roadway centerline to measure the radius of curvature and curve length. The road coordinates are determined by referencing to Test Site 3 as stated in [9]. Ellipses will be constructed, and the radius of curvature can be calculated by using (2.6):

$$R = \frac{[(x')^2 + (y')^2]^{\frac{3}{2}}}{|x'y'' - y'x''|} \tag{2.6}$$

where $x(z)$ and $y(z)$ are from Equation (2.1).

Test site 3 as shown in Fig. 1(a) is located in Interstate 35 (I-35) in Kansas, United States that begins at 39°02'20.43" N, 94°40'26.76" W and ends at 39°02'29.26" N, 94°40'22.67" W with the length of 324 m. The radius of curvature obtained from field measurement is 104.15 m [9]. Coordinates of 9 points along Test Site 3 from Google Maps were chosen and presented in Table 1, the best curve from minimization procedure is fitted as shown in Fig. 1(b) and Fig. 1(c) displayed the fitting of a full ellipse.

We can observe that the centre of the ellipse segment is in between point 4 and point 5 and the radius of curvature in between those points lies between 123.686 m and 94.6137 m. The radius of curvature obtained from our proposed algorithm is nearly equal to the radius of curvature found by Luo et al. [9] which is 104.15 m. We do not provide an exact comparison as we are uncertain of which specific point is referenced in [9]. The method used in our paper shows a high similarity to the field measurement in which radius of curvature obtained from the least squares fitting on a segment of an ellipse is found to be approximately equal to the radius of curvature obtained by Luo et al. [9].

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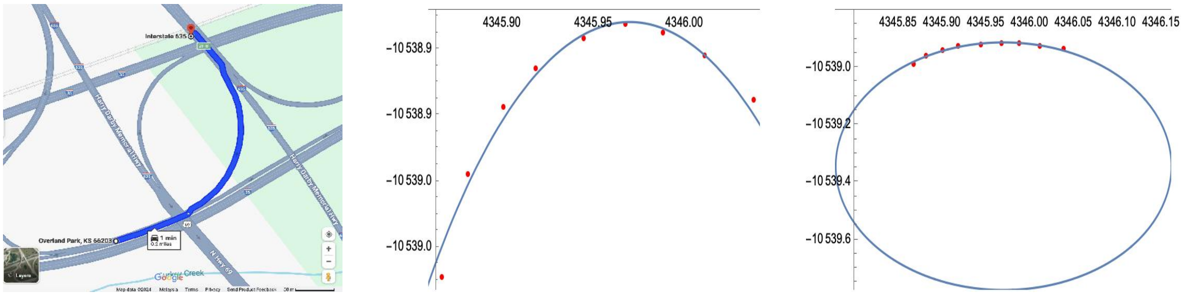

Fig. 1: Test Site 3: (a) Location on Google Maps, (b) 9 points chosen along Test Site 3, and (c) Fitting of a full ellipse on Test Site 3.

Points	Coordinates	Radius of curvature (m)
1	(39.039408, -94.672916)	293.312
2	(39.039537, -94.672637)	213.213
3	(39.039712, -94.672455)	157.344
4	(39.039871, -94.672347)	123.686
5	(39.040104, -94.672268)	94.6137
6	(39.040312, -94.672229)	85.0239
7	(39.040496, -94.672251)	89.7698
8	(39.040704, -94.672315)	110.282
9	(39.040944, -94.672433)	153.020

Tab. 1: The coordinates and radius of curvature for 9 points taken along Test Site 3.

Conclusions:

In this paper, least squares fitting is applied to obtain the best curve fitting to the given data that form a segment of ellipse by minimizing the sum of square errors using simulated annealing. It can be observed that the solution to the minimization problem approximates the data closely by the ellipse. In addition, we fit the data of a small segment of a road to obtain its curvature at any specific point on the road. For perspective, this can be extended in future research for travel time prediction in [15] or for the purpose of road safety in [10]. The positive aspect of our approach lies in its cost efficiency as we rely on the readily available GPS data. Generally, if a set of a parametric data is assumed to behave in ellipse shape, we should be able to perform least squares fitting using the proposed algorithm. A few segment of roads have been tested by using this approach and the results demonstrated were proven reliable by the proposed algorithm in obtaining the radius of curvature. Hence, it can be utilized in real-life applications.

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