Title:
Application of the Shape Uniqueness Theorem to the H-Bézier Curve

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Introduction:

The shape uniqueness theorem for free-form curves shows the conditions on which the shapes of two parametric curves defined by three control points are identical although their parametrization may be different [1]. According to this theorem, even though their blending functions look different, the curves become identical by reparametrization under some conditions on their blending functions.

A lot of researches have been done on the blending functions of free-form curves so far and many types of free-from curves are available for curve designers. These designers must be confused on which type of curve should be used for their design. We hope that the shape uniqueness theorem for free-form curves will help the designers classify and categorize types of curves and select the most suitable one for their design purposes because it identifies the curves which superficially look different but represent the same shape.

In this paper we will apply the shape uniquness therem to the H-Bézier curve [2], whose blending functions are defined by recursively using integral forms. We think that it is worth while to apply the shape uniquness theorem to a uniquely defined free-form curve.

H-Bézier Curve [2, 3]

The H-Bézier curve of degree n with parameter α is

$$\boldsymbol{q}(t) = \sum_{i=0}^{n} Z_i^n(t)\boldsymbol{b}_i \tag{2.1}$$

for $t \in [0, 1]$, where Z_i^n is *the H-basis function* of degree n defined by

$$Z_0^1(t) = \frac{\sinh \alpha(1-t)}{\sinh \alpha},\ Z_1^1(t) = \frac{\sinh \alpha t}{\sinh \alpha} \tag{2.2}$$

and recursively

$$W_i^n(t) = \int_0^t Z_i^n(s)ds \quad (0 \le i \le n) \tag{2.3}$$

$$Z_0^{n+1}(t) = 1 - \frac{W_0^n(t)}{W_0^n(1)} \tag{2.4}$$

$$Z_i^{n+1}(t) = \frac{W_{i-1}^n(t)}{W_{i-1}^n(1)} - \frac{W_i^n(t)}{W_i^n(1)} \quad (1 \le i \le n) \tag{2.5}$$

$$Z_{n+1}^{n+1}(t) = \frac{W_n^n(t)}{W_n^n(1)} \tag{2.6}$$

for $n \ge 1$. The quadratic H-basis functions are

$$Z_0^2(t) = \frac{1-\cosh\alpha(1-t)}{1-\cosh\alpha},\ Z_1^2(t) = \frac{\cosh\alpha(1-t) - \cosh\alpha - 1 + \cosh\alpha t}{1-\cosh\alpha},\ Z_2^2(t) = \frac{1-\cosh\alpha t}{1-\cosh\alpha} \tag{2.7}$$

and the cubic H-basis functions are

$$Z_0^3(t) = \frac{\alpha(1-t) - \sinh\alpha(1-t)}{\alpha - \sinh\alpha} \tag{2.8}$$

$$Z_1^3(t) = \frac{\alpha t + \sinh\alpha(1-t) - \sinh\alpha}{\alpha - \sinh\alpha} - \frac{\sinh\alpha(1-t) + \alpha t\cosh\alpha + \alpha t - \sinh\alpha t - \sinh\alpha}{\alpha\cosh\alpha + \alpha - 2\sinh\alpha} \tag{2.9}$$

$$Z_1^3(t) = \frac{\sinh\alpha(1-t) + \alpha t\cosh\alpha + \alpha t - \sinh\alpha t - \sinh\alpha}{\alpha\cosh\alpha + \alpha - 2\sinh\alpha} - \frac{\alpha(1-t) - \sinh\alpha t}{\alpha - \sinh\alpha} \tag{2.10}$$

$$Z_0^3(t) = \frac{\alpha(1-t) - \sinh\alpha t}{\alpha - \sinh\alpha} \tag{2.11}$$

Note that

$$\frac{Z_1^2(t)^2}{Z_0^2(t)Z_2^2(t)} = 2(1+\cosh\alpha). \tag{2.12}$$

The quadratic rational Bézier basis functions $R_i^2(t)$, $i = 0, 1, 2$ are given by

$$R_0^2(t) = \frac{(1-t)^2}{(1-t)^2 + 2(1-t)tw + t^2} \tag{2.13}$$

$$R_1^2(t) = \frac{2(1-t)tw}{(1-t)^2 + 2(1-t)tw + t^2} \tag{2.14}$$

$$R_2^2(t) = \frac{t^2}{(1-t)^2 + 2(1-t)tw + t^2} \tag{2.15}$$

where w is the weight of the second control point. Then

$$\frac{R_1^2(t)^2}{R_0^2(t)R_2^2(t)} = 4w^2 \tag{2.16}$$

From the shape uniqueness theorem for the free-form curve defined by three control points [1], the shapes of quadratic H-Bézier and rational Bézier curves are identical for the same given control points if $2(1+\cosh\alpha) = 4w_h^2$, i.e. the equivalent weight $w_h = \sqrt{\frac{1+\cosh\alpha}{2}}$. Since $\cosh\alpha > 1$ for $\alpha > 0$, $\sqrt{\frac{1+\cosh\alpha}{2}} > 1$.

Shape Equivalence of the cubic H-Bézier Curve

In this section, we will discuss on the shapes of cubic H-Bézier curve and the up-degreed curve of the quadratic rational Bézier curve based on the recursive algorithm explained the previous section. The up-degree procedure is different from the generation by multiplying $(1-t)+t$ to the lower basis functions. In the quadratic rational Bézier basis functions are up-degreed as

$$R_0^3(t) = \frac{\sqrt{-w-1}\sqrt{w-1}(\log(2t(t(-w)+t+w-1)+1)-2t+2)}{2\sqrt{-w-1}\sqrt{w-1}+4w\tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right)} + \frac{2w\left(\tan^{-1}\left(\frac{(1-2t)\sqrt{w-1}}{\sqrt{-w-1}}\right)+\tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right)\right)}{2\sqrt{-w-1}\sqrt{w-1}+4w\tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right)} \tag{2.17}$$

$$R_1^3(t) = \frac{(w-1)\left((w+1)\log(2t(t(-w)+t+w-1)+1)+2\sqrt{-w-1}\sqrt{w-1}\tan^{-1}\left(\frac{(2t-1)\sqrt{w-1}}{\sqrt{-w-1}}\right)\right)}{2\left(\sqrt{-w-1}\sqrt{w-1}+2\tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right)\right)\left(\sqrt{-w-1}\sqrt{w-1}+2w\tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right)\right)} - \frac{2\sqrt{-w-1}\sqrt{w-1}\tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right)((2t-1)(w-1)+\log(2t(t(-w)+t+w-1)+1))}{2\left(\sqrt{-w-1}\sqrt{w-1}+2\tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right)\right)\left(\sqrt{-w-1}\sqrt{w-1}+2w\tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right)\right)} \tag{2.18}$$

$$R_2^3(t) = \frac{(w-1)\left((w+1)\log(2t(t(-w)+t+w-1)+1)-2\sqrt{-w-1}\sqrt{w-1}\tan^{-1}\left(\frac{(2t-1)\sqrt{w-1}}{\sqrt{-w-1}}\right)\right)}{2\left(\sqrt{-w-1}\sqrt{w-1}+2\tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right)\right)\left(\sqrt{-w-1}\sqrt{w-1}+2w\tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right)\right)} + \frac{2\sqrt{-w-1}\sqrt{w-1}\tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right)((2t-1)(w-1)-\log(2t(t(-w)+t+w-1)+1))}{2\left(\sqrt{-w-1}\sqrt{w-1}+2\tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right)\right)\left(\sqrt{-w-1}\sqrt{w-1}+2w\tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right)\right)}$$

$$R_3^3(t) = \frac{\sqrt{-w-1}\sqrt{w-1}(\log(2t(t(-w)+t+w-1)+1)+2t)}{2\sqrt{-w-1}\sqrt{w-1}+4w\tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right)} + \frac{2w\left(\tan^{-1}\left(\frac{(2t-1)\sqrt{w-1}}{\sqrt{-w-1}}\right)+\tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right)\right)}{2\sqrt{-w-1}\sqrt{w-1}+4w\tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right)} \tag{2.19}$$

These basis functions are different from those of the cubic Bézier curve and, for example $R_1^3(t)^2/(R_0^3(t)R_2^3(t))$ is dependent on parameter t. Hence they are quite different from those of the cubic rational Bézier curve. Please refer to [5] on the shape uniqueness theorem for the free-form crve defined by four or more control points. Figure 1 shows these basis functions with $w = 1/2$.

Reparametrization of Integration

Identical shape of two parametric curves is defined as follows [4]:

Definition 1 *For two parametric curves $\boldsymbol{r} : I \to R^3$ and $\tilde{\boldsymbol{r}} : \tilde{I} \to R^3$, there exists a C^∞ function $\phi : I \to \tilde{I}$,* 1) *ϕ is a one to one and onto mapping from I to $\tilde{I}$.* 2) *ϕ is strictly increasing.* 3) *For all $t \in I$, $\tilde{\boldsymbol{r}}(\phi(t)) = \boldsymbol{r}(t)$. We say that $\boldsymbol{r}$ and $\tilde{\boldsymbol{r}}$ define the same curve or their shapes are identical.*

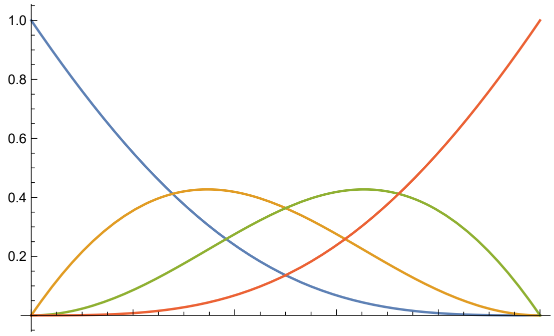

Fig. 1: Basis functions of the up-degreed rational quadratic Bézier curve with $w = 1/2$.

Then $\tilde{\boldsymbol{r}}((\phi(t))$ is called reparametrization of $\boldsymbol{r}(t)$.

For example, there is a function $\phi(t)$ such that

$$R_i^2(\phi(t)) = Z_i^2(t) \quad i = 0, 1, 2 \tag{2.20}$$

We assume that

$$\int_0^{\phi(t)} R_0^2(\phi(t))d\phi(t) = T_0^3(\phi(t)) \tag{2.21}$$

Since $\phi(1) = 1$,

$$R_0^3(\phi(t)) = 1 - \frac{T_0^3(\phi(t))}{T_0^3(1)},\ R_1^3(\phi(t)) = \frac{T_0^3(\phi(t))}{T_0^3(1)} - \frac{T_1^3(\phi(t))}{T_1^3(1)}, \tag{2.22}$$

$$R_2^3(\phi(t)) = \frac{T_1^3(\phi(t))}{T_1^3(1)} - \frac{T_2^3(\phi(t))}{T_2^3(1)},\ R_3^3(\phi(t)) = \frac{T_2^3(\phi(t))}{T_2^3(1)}. \tag{2.23}$$

Hence, R_i^3 is naturally reparametrized by $\phi(t)$.

Before discussing the relationships among Z_i^3 and R_i^3, $i = 0, \cdots, 3$, we show the properties of Z_i^3. From their definition,

$$\sum_{i=0}^{3} Z_i^3(t) = 1,\ \sum_{i=0}^{3} \frac{dZ_i^3(t)}{dt} = 0,\ \sum_{i=0}^{3} \int_0^t Z_i^3(t)dt = t. \tag{2.24}$$

Especially when $t = 1$, we obtain

$$\sum_{i=0}^{3} \int_0^1 Z_i^3(t)dt = 1 \tag{2.25}$$

$Z_i^2(t)$ are given by

$$Z_0^2(t) = -\frac{dZ_0^3(t)}{dt} \int_0^1 Z_0^2(t)dt \tag{2.26}$$

$$Z_1^2(t) = \frac{dZ_0^3(t)}{dt} \int_0^1 Z_0^2(t)dt - \frac{dZ_1^3(t)}{dt} \int_0^1 Z_1^2(t)dt \tag{2.27}$$

$$Z_2^2(t) = \frac{dZ_1^3(t)}{dt} \int_0^1 Z_1^2(t)dt - \frac{dZ_2^3(t)}{dt} \int_2^1 Z_2^2(t)dt = \frac{dZ_3^3(t)}{dt} \int_2^1 Z_2^2(t)dt \tag{2.28}$$

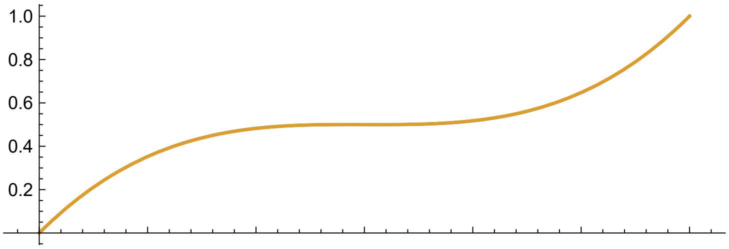

Fig. 2: The identical two curves with $\alpha = 1/2$ and $w = \sqrt{\frac{1+\cosh\frac{1}{2}}{2}}$.

Hence

$$\frac{Z_1^2(t)^2}{Z_0^2(t)Z_2^2(t)} = -\frac{(\frac{dZ_0^3(t)}{dt}\int_0^1 Z_0^2(t)dt - \frac{dZ_1^3(t)}{dt}\int_0^1 Z_1^2(t)dt)^2}{\frac{dZ_0^3(t)}{dt}\int_0^1 Z_0^2(t)dt\frac{dZ_3^3(t)}{dt}\int_2^1 Z_2^2(t)dt} \tag{2.29}$$

is not dependent on parameter t and a constant.

Since R_i^3 is naturally reparameterized by $\phi(t)$, given the same four control points, the shapes of the curves whose basis functions are $Z_i^3(t)$ and $R_i^3(t)$, respectively are identical. Figure 2 shows the identical two curves defined with the same control points $(0,0)$, $(1,1)$, $(2,0)$, and $(3,1)$.

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